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OPTIMAL DESIGN OF IMPULSIVELY LOADED PLASTIC BEAMS FOR ASYMMETRIC MODE MOTIONS

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Abstract—Optimal design of a rigid-plastic stepped beam is discussed assuming the mode form of motion. Such beam dimensions are sought for which a minimum of local or mean deflection is attained within designs of constant volume. It is assumed that the prescribed kinetic energy is imparted to the structure at the initial instant with free motion occurring afterwards. It is shown that besides three symmetric modes of motion, also the asymmetric modes may exist. An optimal design for asymmetric modes is determined and compared with a respective design for symmetric modes, obtained previously in [1].

1. INTRODUCTION

In the previous paper by Lepik and Mróz[1], the problem of optimal design of plastic beams and plates was discussed for the case of impulsive and pressure loading. It was assumed, that at the initial instant the structure attains the kinetic energy K_0 through impulsive loading and the subsequent motion proceeds to the modal form $w(x, t) = W(x)\Phi(t)$; here $w(x, t)$ denotes the lateral beam deflection. Such optimal mode forms and beam dimensions were sought which correspond to minimum of local or mean deflection. The analysis of optimal design for pressure loading indicates that the one-degree-of-freedom modes are essential in design since they are permanent forms of motion and such modes were considered in optimizing design for the case of impulsive loading.

It is natural to expect that for loading symmetrically distributed with respect to the beam centre, the corresponding motion will also be symmetric and only symmetric modes need to be considered. However, when we assume that only initial kinetic energy K_0 or momentum P_0 imparted to the structure by the impulsive loading is specified and no restriction is imposed on the initial pressure distribution corresponding to this impulse, it may be expected that asymmetric modes can also occur for a symmetrically designed structure. It is the aim of this paper to investigate the asymmetric mode motions and the respective optimal designs for such asymmetric modes. Thus the present analysis is pertinent to such cases when only *limited information* exists on possible pressure distribution but the initial kinetic energy K_0 or momentum P_0 attained by the structure can be evaluated more precisely. In such cases, we should consider all kinematically and dynamically admissible modes and provide proper designs satisfying constraints set on maximal or mean deflections. In this work, we shall compare symmetric and asymmetric mode solutions corresponding to the same initial kinetic energy imparted to the structure. A related problem of mode solutions based on the same total pressure impulse or initial momentum is not discussed here and will be considered separately. We also do not impose constraints on accelerations of motion which are important in considering shock or vibration damping in vehicle structures.

The basic equations are derived in Section 2 and in Section 3 the kinematically and dynamically admissible mode motions are investigated. Optimal design and numerical results are discussed in Section 4.

2. BASIC EQUATIONS

Let us consider a rigid plastic beam of rectangular cross section with segmentwise constant thickness; it is simply supported at both ends (Fig. 1a). The beam is loaded impulsively, the kinetic energy at the initial instant K_0 is given.

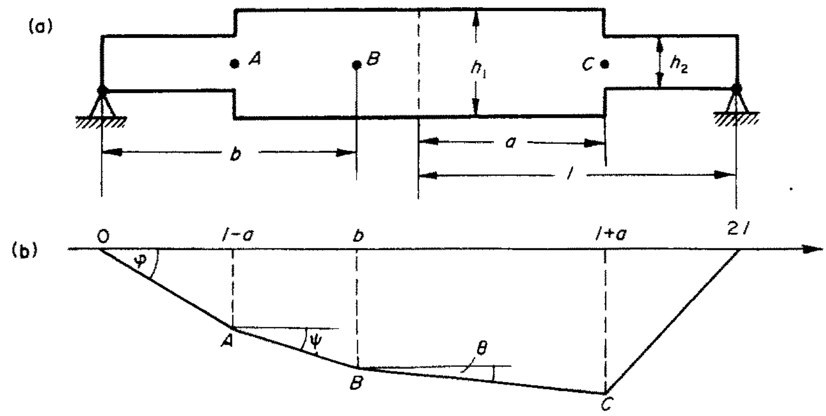

Fig. 1. (a) Beam dimensions. (b) Yield mechanism with plastic hinges at A, B and C.

In the case of mode solutions, only stationary plastic hinges at points A, B and C can appear (Fig. 1b). Let us denote the angles between the beam segments and the horizontal direction by φ, ψ, θ.

$$\xi=\frac{x}{l},\quad \alpha=\frac{a}{l},\quad \beta=\frac{b}{l},\quad \lambda=\frac{\psi}{\varphi},\quad \mu=\frac{\theta}{\varphi} \tag{1}$$

we can describe the deflection field as follows

$$W(\xi)=\frac{w(\xi,t)}{l\varphi(t)}=\begin{cases}\xi & \text{for } \xi\in[0,1-\alpha]\\ 1-\alpha+(\xi-1+\alpha)\lambda & \text{for } \xi\in[1-\alpha,\beta]\\ 1-\alpha(\beta-1+\alpha)\lambda+(\xi-\beta)\mu & \text{for } \xi\in[\beta,1+\alpha]\\ \dfrac{2-\xi}{1-\alpha}[1-\alpha+(\beta-1+\alpha)\lambda+(1+\alpha-\beta)\mu] & \text{for } \xi\in[1+\alpha,2].\end{cases} \tag{2}$$

The dimensionless quantity β determines the cross section position where the bending moment M has a maximal value. Since the plastic work dissipated at hinges A, B and C must be positive and the bending moment M cannot reach negative values, the diagram $w=w(\xi,t)$ must be convex. This requirement puts the following restrictions on the quantities λ and μ:

$$\mu\le\lambda\le1,\quad (\alpha+\beta-1)\lambda+(2-\beta)\mu+1-\alpha\ge0. \tag{3}$$

The equations of motion are

$$\frac{dM}{d\xi}=lQ^s,\quad \frac{dQ^s}{d\xi}=\rho lDh(\xi)\ddot{w}, \tag{4}$$

where M and Q^s are the bending moment and the shear force, ρ denotes the density, D and h denote the beam width and height. Since in the case of modal solutions the quantities M and Q^s do not depend upon t, it follows from the eqns (2) and (4) that $\ddot{\varphi}=\text{constant}$.

Now let us integrate the eqns (4). The integration constants are calculated from the following conditions: (1) boundary conditions $M(0)=M(2)=0$; (2) continuity conditions for M and Q^s at $\xi=1\pm\alpha$; (3) the extremum condition $dM/d\xi=0$ at $\xi=\beta$; (4) continuity condition for M at $\xi=\beta$.

Carrying out all these calculations one obtains

$$M(1-\alpha)=-\frac{\rho}{6}Dh_2l^3\ddot{\varphi}(A_1+A_2\lambda)$$

$$M(\beta)=-\frac{\rho}{6}Dh_2l^3\ddot{\varphi}(B_1+B_2\lambda)$$

$$M(1+\alpha) = -\frac{\rho}{6} Dh_2 l^3 \ddot{\varphi}(C_1 + C_2\lambda + C_3\mu)$$

$$M(2) = -\frac{\rho}{6} Dh_2 l^3 \ddot{\varphi}(D_1 + D_2\lambda + D_3\mu) \tag{5}$$

where

$$\begin{aligned}
A_1 &= 2(1-\alpha)^2[1-\alpha+3\gamma(\alpha+\beta-1)] \\
A_2 &= 3\gamma(1-\alpha)(\alpha+\beta-1)^2 \\
B_1 &= (1-\alpha)[2(1-\alpha)^2+3\gamma(\alpha+\beta-1)(1-\alpha+\beta)] \\
B_2 &= \gamma(1+2\beta-\alpha)(\alpha+\beta-1)^2 \\
C_1 &= 2(1-\alpha)[(1-\alpha)^2-3\gamma(1+\alpha^2-\beta-\alpha\beta)] \\
C_2 &= \gamma(\alpha+\beta-1)[(1+2\beta-\alpha)(\alpha+\beta-1)-3(1+\alpha-\beta)^2] \\
C_3 &= -\gamma(1+\alpha-\beta)^3 \\
D_1 &= -12\gamma(1-\alpha)(1-\beta) \\
D_2 &= -(\alpha+\beta-1)[2(1-\alpha)^2+\gamma(10+4\alpha-11\beta-2\alpha^2-\alpha\beta+\beta^2)] \\
D_3 &= -(1+\alpha-\beta)[2(1-\alpha)^2+\gamma(4+2\alpha-5\beta+\alpha\beta-2\alpha^2+\beta^2)].
\end{aligned} \tag{6}$$

The bending moments M must satisfy the following inequalities

$$M(1-\alpha) = M_A \leq \frac{1}{4} D\sigma_0 h_2^2, \quad M(\beta) = M_B \leq \frac{1}{4} D\sigma_0 h_1^2, \quad M(1+\alpha) = M_C \leq \frac{1}{4} D\sigma_0 h_2^2. \tag{7}$$

In these formulae σ_0 denotes the yield stress of the material.

In the following, only designs of the same volume will be composed. Since $V = 2Dh_2 l\Delta$, where $\Delta = 1-\alpha+\alpha\gamma$ and $\gamma = h_1/h_2$, there is

$$h_1 = \frac{V\gamma}{2Dl\Delta}, \quad h_2 = \frac{V}{2Dl\Delta}. \tag{8}$$

In order to present the basic equations in a dimensionless form, let us introduce the quantities

$$N = \frac{3\sigma_0 V}{4\rho l^4 D}, \quad P = -\frac{\ddot{\varphi}}{N}, \quad Q = -\frac{\ddot{\psi}}{N}, \quad R = -\frac{\ddot{\theta}}{N}. \tag{9}$$

Now the inequalities (7) take the form

$$\begin{aligned}
A_1 P + A_2 Q &\leq 1/\Delta \\
B_1 P + B_2 Q &\leq \gamma^2/\Delta \\
C_1 P + C_2 Q + C_3 R &\leq 1/\Delta
\end{aligned} \tag{10}$$

and the boundary condition $M(2) = 0$ gives

$$D_1 P + D_2 Q + D_3 R = 0. \tag{11}$$

Now, let us find the expression for the residual deflections. If the motion ends at the time $t = t_f$, we have $\dot{\varphi}(t_f) = 0$. Integrating the equation $\ddot{\varphi} = \text{constant}$, one obtains

$$\varphi(t_f) = -\frac{1}{2}\frac{\dot{\varphi}^2(0)}{\ddot{\varphi}}. \tag{12}$$

The expression on the r.h.s. of eqn (12) will be calculated by using the energy balance equation $\dot{K}+\dot{A}=0$, where K is the kinetic energy and A is the plastic work.

The quantities $\dot{K}$ and $\dot{A}$ can be calculated from the formulae

$$\dot{K}=\frac{2K_0}{\dot{\varphi}^2(0)}\dot{\varphi}(t)\ddot{\varphi}$$

$$\dot{A}=M_A(\dot{\varphi}+\dot{\psi})+M_B(\dot{\psi}-\dot{\theta})+M_C\left[\dot{\theta}+\frac{1-\alpha+(\alpha+\beta-1)\lambda+(1+\alpha-\beta)\mu}{1-\alpha}\dot{\varphi}\right].$$

Making use of formulae (7) and (9), we obtain

$$-\frac{\dot{\varphi}^2(0)}{\ddot{\varphi}}=\frac{32K_0Dl^2}{\sigma_0V^2}\frac{P\Delta^2}{T}, \tag{13}$$

where

$$T=2P+\gamma^2(Q-R)+\frac{1}{1-\alpha}[(2\alpha+\beta-2)Q+(2-\beta)R]. \tag{14}$$

The residual deflection can be presented in the form

$$w(\xi,t_f)=lW(\xi)\varphi(t_f), \tag{15}$$

where the function $W(\xi)$ is defined by the formula (2). In view of (12) and (13), the expression (15) can be written in the form

$$w(\xi,t_f)=\frac{16K_0Dl^3}{\sigma_0V^2}\frac{W(\xi)P\Delta^2}{T}. \tag{16}$$

Different objective functions can be proposed for solving the optimal design problem. Firstly, we can minimize the maximal value of the final deflection:

$$W_{max}=\max\,[W(1-\alpha),\quad W(\beta),\quad W(1+\alpha)]=\min.$$

It will be convenient to present the final deflection in a dimensionless form

$$F_1=\frac{\max\limits_{\xi} w(\xi,t_f)}{\max\limits_{\xi} w_*(\xi,t_f)} \tag{17}$$

where $w_*(\xi,t_f)$ is the residual deflection for a beam of constant thickness. Now it follows from the formula (16) that

$$F_1=2W_{max}\frac{P\Delta^2}{T}=\min. \tag{18}$$

For a uni-directional impulse, also the mean deflection

$$\bar{w}=\int_0^1 w(\xi,t_f)\,d\xi$$

can be assumed as a meaningful objective function. In this case, we shall have

$$F_2=\frac{\Delta^2}{T}[2(1-\alpha^2)P+(\alpha+\beta-1)(2+2\alpha-\beta)Q+(2-\beta)(1+\alpha-\beta)R]=\min. \tag{19}$$

The next possibility is to minimize the quadratic weighted mean deflection

$$\tilde{w} = \left(l \int_0^1 w^2(\xi, t_f) h(\xi)\, d\xi \right)^{1/2}. \tag{20}$$

This leads us to the following optimization problem

$$F_3 = \frac{\sqrt{6}\Delta^{3/2} S}{T} = \min \tag{21}$$

where

$$S = (S_1 + S_2 + S_3 + S_4)^{1/2}$$

$$S_1 = \frac{1}{3}(1-\alpha)^3 P^2$$

$$S_2 = \gamma(\alpha+\beta-1)[(1-\alpha)^2 P^2 + (1-\alpha)(\alpha+\beta-1)PQ + \frac{1}{3}(\alpha+\beta-1)^2 Q^2]$$

$$S_3 = \gamma(1+\alpha-\beta)\{[(1-\alpha)P + (\alpha+\beta-1)Q]^2 + (1+\alpha-\beta) \times [(1-\alpha)P + (\alpha+\beta-1)Q]R + \frac{1}{3}(1+\alpha-\beta)^2 R^2\}$$

$$S_4 = \frac{1}{3}(1-\alpha)[(1-\alpha)P + (\alpha+\beta-1)Q + (1+\alpha-\beta)R]^2. \tag{22}$$

If we set $\gamma = 1$ in the formulae (22), the following quantity

$$\tilde{w} = \left(l \int_0^1 w^2\, d\xi \right)^{1/2} \tag{23}$$

should be minimized. For this case, the value of the function F defined by (21) is denoted by F_4.

3. ASYMMETRIC MODES OF BEAM MOTION

Now, we shall examine which asymmetric modes of beam motion are possible.

Case 1. Let us begin from the case, where a plastic hinge appears at the point $\xi = \beta$. It follows from Fig. 2(a) that $\lambda = 1$, $\mu = -\beta/(2-\beta)$ and consequently, $P = Q$, $R = \mu Q$. Substituting these values in (11), we obtain an equation for the parameter β. Carrying out the calculations we obtain

$$\beta_1 = 1; \quad \beta_{2,3} = 1 \pm \left(1 + \frac{1}{2\gamma}(\gamma-1)(1-\alpha)^3\right)^{1/2}. \tag{24}$$

Since the values β_2 and β_3 do not belong to the interval $[0, 2]$ and $\beta = 1$ corresponds to the symmetric case, we have proved that the asymmetric mode is not possible in Case 1.

Case 2. Now, a plastic hinge appears at $\xi = 1-\alpha$ and; the sections $\xi = \beta$ and $\xi = 1+\alpha$ remain rigid. It follows from Fig. 2(b) that

$$\frac{w_A}{w_B} = \frac{1+\alpha}{2-\alpha}; \quad \frac{w_A}{w_C} = \frac{1+\alpha}{1-\alpha}.$$

Making use of the eqns (2) we have

$$\lambda = \mu = -\frac{1-\alpha}{1+\alpha}, \quad Q = R = \lambda P.$$

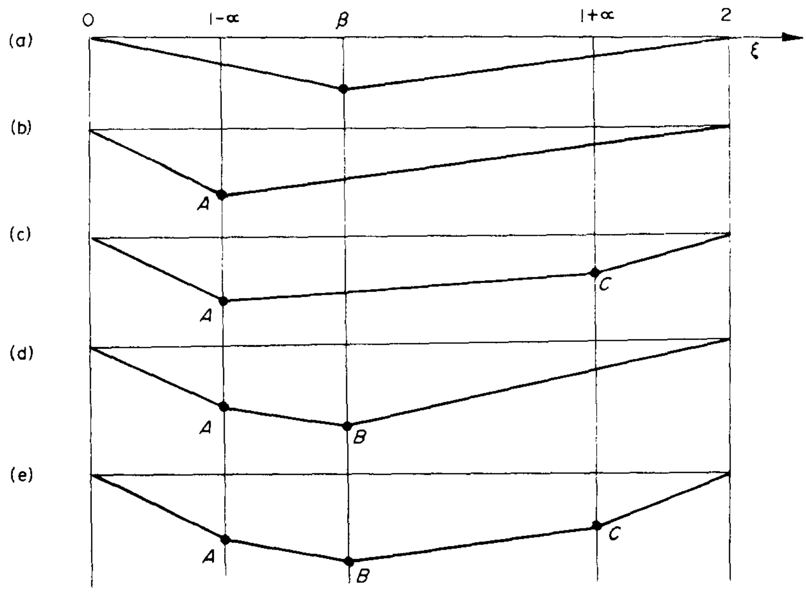

Fig. 2. Different mode forms.

Equation (11) takes the form

$$(1+\alpha)D_1-(1-\alpha)(D_2+D_3)=0$$

and solving it with respect to β, we have

$$\beta_{1,2}=2\pm\left(\frac{2\alpha}{3\gamma}(1-\alpha)^2+1+\alpha^2-\frac{2}{3}\alpha^3\right)^{1/2} \tag{25}$$

The sign "+" before the square root does not fit our solution since $\beta_1>2$. The other root β_2 is suitable, since it always belongs to the interval $[1-\alpha, 1+\alpha]$.

The value of P can be calculated from the first formula of (10) after replacing the inequality sign by an equality. This case occurs when the second and third expressions of (10) are satisfied as strong inequalities.

The case, where a plastic hinge appears at $\xi=1+\alpha$ and the cross sections $\xi=1-\alpha$ and $\xi=\beta$ are rigid, can be discussed in an analogous way.

Case 3. In this case the plastic hinges occur at the points A and C (Fig. 2c). Since the section B remains rigid, we have $\lambda=\mu$ or $Q=R$. For determining the two quantities P and Q we have now three equations; (1) the first and third equations of (10), replacing there the inequalities by equalities and (2) eqn (11). Therefore, the following condition is obtained

$$\begin{bmatrix} A_1 & A_2 & 1/\Delta \\ C_1 & C_2+C_3 & 1/\Delta \\ D_1 & D_2+D_3 & 0 \end{bmatrix}=0. \tag{26}$$

It can be shown that this condition is fulfilled only when $\beta=1$; consequently, also in this case only a symmetric mode can occur.

Case 4. Now let us analyse the case, where plastic hinges appear at points A and B (Fig. 2d). The two first inequalities of (10) and the last inequality of (3) are satisfied as equalities, besides the eqn (11) is valid. So we have four equations for three quantities P, Q, R; thus, the determinant of the enlarged system must be zero. This condition provides the following equation

$$[(\alpha+\beta-1)D_3-(2-\beta)D_2](B_1-\gamma^2A_1)+[(2-\beta)D_1-(1-\alpha)D_3](B_2-\gamma^2A_2)=0 \tag{27}$$

from which the quantity β can be calculated. The values of P, Q and R are calculated from the formulae

$$P=\frac{B_2-\gamma^2A_2}{\Delta(A_1B_2-A_2B_1)},\quad Q=\frac{\gamma^2A_1-B_1}{\Delta(A_1B_2-A_2B_1)},\quad R=-\frac{1}{D_3}(D_1P+D_2Q). \tag{28}$$

This solution is valid, if: (1) the third equation of (10) is satisfied as a strong inequality, and (2) $R\le Q\le P$.

The case when the plastic hinges appear at the cross sections B and C can be treated analogously.

Case 5. Here we have three plastic hinges-at the cross sections A, B and C. Now, all three weak inequalities (10) are satisfied as equalities, and the eqn (11) must be fulfilled. Thus, we obtain again a system of four equations for P, Q and R. This system has a solution only when the determinant of the enlarged system is zero. It can be shown that this requirement is fulfilled only for $\beta = 1$. It is thus seen that also in this case an asymmetric mode cannot exist.

It has been shown that all five modes of motion are kinematically and dynamically admissible; from these three modes are symmetric (1,3,5) and two asymmetric (2,4).

4. DISCUSSION OF NUMERICAL RESULTS

Numerical calculations were carried out for different values of α and γ and the results are presented in Figs. 3–5. Different yield mechanisms in the plane (α, γ) are shown in Fig. 3; the respective numbers correspond to particular modes of Fig. 2. It is seen that for a sufficiently thin central portion only mode 1 occurs. For increasing γ, the symmetric modes 1,3,5 occur

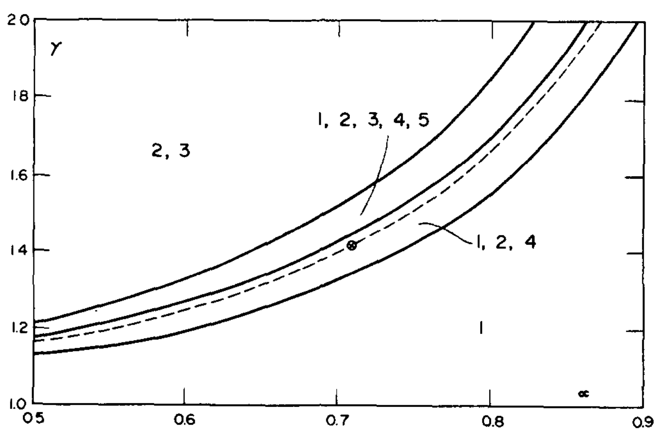

Fig. 3. Occurrence of modes 1–5 in the plane (α, γ). Optimal design is marked by $\otimes$.

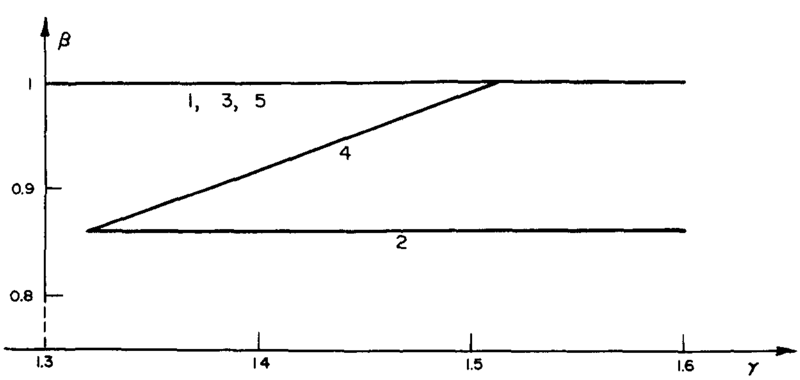

Fig. 4. Dependence of β on γ for $\alpha = 0.7$ for the mode motions 1–5.

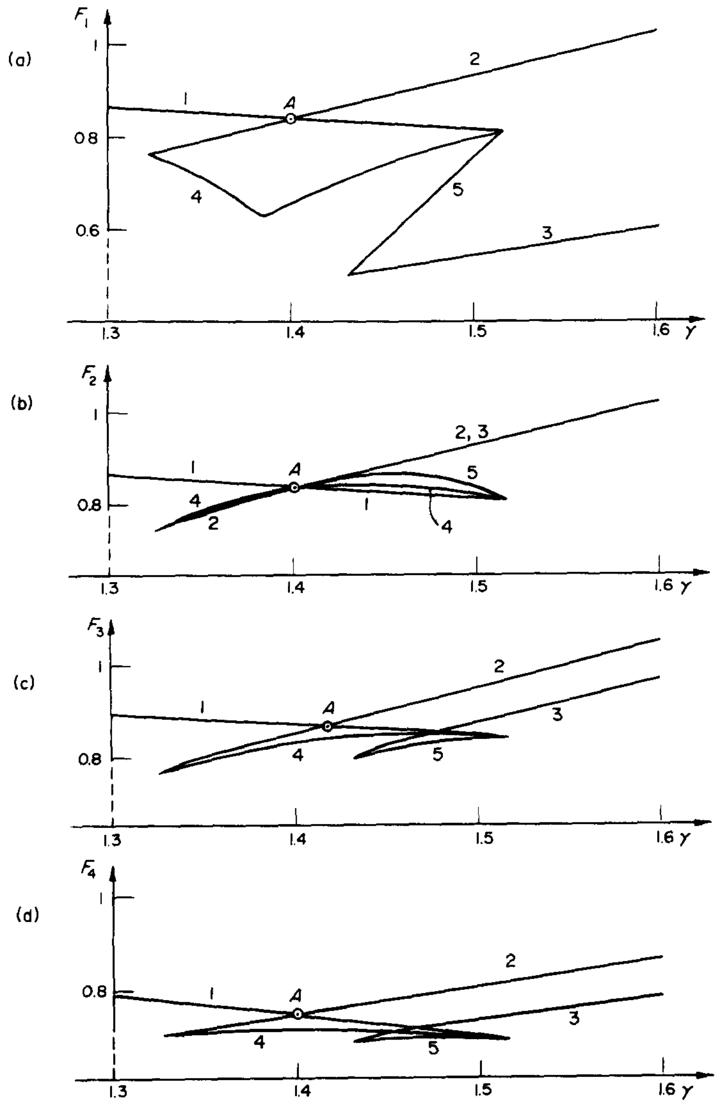

Fig. 5. Dependence of dimensionless deflections (for four objective functions) on γ for $\alpha = 0.7$.

together with the asymmetric modes 2,4. The following four combinations of modes were found: 1; 1,2,4; 2,3; and 1,2,3,4,5. The combinations 1,2,4 and 1,2,3,4,5 may exist only in a narrow zone of the plane (α, γ).

As it was shown in Section 2, the bending moment has its maximum at $\xi = \beta$. Fig. 4 shows how the parameter β depends upon γ for $\alpha = 0.7$. Numbers in this diagram indicate the respective cases of motion (for the cases 1, 3, and 5, we have $\beta = 1$).

Dependence of the dimensionless deflection on γ for $\alpha = 0.7$ is shown in Fig. 5. Calculations were carried out for four different objective functions which were defined in Section 2; the corresponding values of dimensionless deflections are denoted by F_1, F_2, F_3 and F_4. From these diagrams the following conclusions can be drawn. When for a specified set of values of α and γ several modes of motion are possible, it indicates that any of the modes may occur in reality when proper loading distribution is applied at the initial instant. Since no information is available on this distribution, the design should be based on such modes which correspond to largest final deflections. Therefore the asymmetric mode 2 should be accounted for since it corresponds to larger deflections than the modes 3–5. Thus the optimal values of α and γ should correspond to intersection of modes 1 and 2 (point A in Fig. 5). It follows from Figs. 5(a)–(d) that the choice of objective function has only small effect on the optimal values of the design parameters α, γ. We shall therefore use the local deflection measure, expressed by (18).

Equating the dimensionless deflections for the modes 1 and 2, it follows from (18) that

$$\gamma = (1-\alpha^2)^{-1/2} \tag{29}$$

and

$$F_1 = [(1-\alpha)(1-\alpha^2)^{1/2} + \alpha]^2. \tag{30}$$

For the optimal solution, there must be $dF_1/d\alpha = 0$ and from (29), (30) we obtain

$$\alpha_{opt} = 1/\sqrt{2} = 0.71, \quad \gamma_{opt} = \sqrt{2} = 1.41, \quad F_{opt} = 0.84. \tag{31}$$

This result differs essentially from the result

$$\alpha_{opt} = 0.9, \quad \gamma_{opt} = 2.6, \quad F_{opt} = 0.60 \tag{32}$$

which was obtained in [1] by considering only symmetric modes. Thus when the possibility of asymmetric loading distribution exists, the design (31) should be used rather than (32). The reduction of deflection with respect to the beam of constant thickness is now 16% whereas this reduction for symmetric mode motion was 40%.

Now, let us calculate the dimensionless deflection F_1 for values of γ and α satisfying (29); the results are presented in Table 1. It is seen that the values of F_1 for $\alpha \in [0.4; 0.9]$ differ from the optimal value $F_1 = 0.836$ only by less than 7%. Thus, for any value of $\alpha \in [0.4; 0.9]$ the parameter γ can be determined from (29) and the design will correspond to the maximal deflection close to that corresponding to optimal design (31). The relation (29) is shown in Fig. 3 by the dotted line.

Table 1. Values of α,γ and of minimal non-dimensional deflection at the intersection of modes 1 and 2

α	γ	F_1
0.4	1.09	0.902
0.5	1.16	0.871
0.6	1.25	0.846
0.7	1.40	0.836
0.8	1.67	0.846
0.9	2.29	0.890

5. CONCLUDING REMARKS

The present paper complements the previous works [1, 2] on optimal design of inelastic structures under impulsive loading by using the concept of mode motion as most essential in representing structural response. It is believed that in many cases the design may be performed by considering only mode forms and neglecting the initial transient period. Such a situation occurs when only limited information is available on the spatial or time distribution of dynamic pressure and only the initial kinetic energy K_0 or momentum P_0 imparted to the structure enters in the design procedure. Usually, the initial transient motion tends to the modal form which predominates in the final period before the rest and provides a dominant contribution to the final deflection. Thus the study of modal forms and their dependence on design parameters constitutes the major problem in this procedure. It turns out that both symmetric and non-symmetric mode forms should be considered even if the design is symmetric with respect to the central axis.

The present analysis also casts light on the effect of *coexistence* of different mode motions which correspond to different eigenforms of a highly nonlinear eigenvalue problem. Note that all considered basic modal forms correspond to the downward motion and for a uniform beam, only one symmetric form could exist. Besides, there is a variety of higher-order modes corresponding to combined downward and upward motion, analogous to higher-order modes in linear elastic beams. Experimental evidence of such higher-order eigenmodes in plastic structures was presented in [3].

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